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# LOGARITHMIC ORDER AND DUAL LOGARITHMIC ORDER

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**Abstract.** We shall define the following four orders for strictly positive operators  $A$  and  $B$  on a Hilbert space  $H$  such that  $1 \notin \sigma(A), \sigma(B)$ .

*Strictly logarithmic order* (denoted by  $A \succ_{sl} B$ ) is defined by  $\frac{A-I}{\log A} > \frac{B-I}{B \log B}$ .

*Logarithmic order* (denoted by  $A \succ_l B$ ) is defined by  $\frac{A-I}{\log A} \geq \frac{B-I}{\log B}$ .

*Strictly dual logarithmic order* (denoted by  $A \succ_{sdl} B$ ) is defined by  $\frac{A \log A}{A-I} > \frac{B \log B}{B-I}$ .

*Dual Logarithmic order* (denoted by  $A \succ_{dl} B$ ) is defined by  $\frac{A \log A}{A-I} \geq \frac{B \log B}{B-I}$ .

Firstly we shall show direct and simplified proofs of operator monotonicity of logarithmic function  $f(t) = \frac{t-1}{\log t}$  and dual logarithmic function  $f^*(t) = \frac{t \log t}{t-1}$ .

In what follows, let  $A$  and  $B$  be strictly positive operators on a Hilbert space  $H$  such that  $1 \notin \sigma(A), \sigma(B)$ . Secondary we shall show the following:

(\*)  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(†)  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

By using these two results (\*) and (†), we summarize the following interesting contrast among  $A > B > 0$ ,  $A \geq B > 0$ ,  $\log A > \log B$  and  $\log A \geq \log B$ .

(l-i)  $A > B > 0 \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(l-ii)  $A \geq B > 0 \implies A^\alpha \succ_l B^\alpha$  for all  $\alpha \in (0, 1]$ .

(l-iii)  $\log A \geq \log B \implies$  for any  $\delta \in (0, 1]$ , there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(l-iv)  $\log A \geq \log B \implies$  for any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^{p\alpha} \succ_l B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .

(dl-i)  $A > B > 0 \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(dl-ii)  $A \geq B > 0 \implies A^\alpha \succ_{dl} B^\alpha$  for all  $\alpha \in (0, 1]$ .

(dl-iii)  $\log A \geq \log B \implies$  for any  $\delta \in (0, 1]$ , there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(dl-iv)  $\log A \geq \log B \implies$  for any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p)^{p\alpha} \succ_{dl} B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .

Finally we cite a counterexample related to (l-iii) and (dl-iii).

## 1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space  $H$ . An operator  $T$  is said to be positive (denoted by  $T \geq 0$ ) if  $(Tx, x) \geq 0$  for all  $x \in H$  and also an operator  $T$  is said to be strictly positive (denoted by  $T > 0$ ) if  $T$  is positive and invertible. The strictly chaotic order is defined by  $\log A > \log B$  for strictly positive operators  $A$  and  $B$ .

It is well known that the usual order  $A \geq B > 0$  ensures the chaotic order  $\log A \geq \log B$  since  $\log t$  is operator monotone function.

Also it is known by [Theorem,6] and [Example 5.1.12 and Corollary 5.1.11, 5] that

$$A \geq B > 0 \text{ ensures } \frac{A - I}{\log A} \geq \frac{B - I}{\log B}$$

and

$$A \geq B > 0 \text{ ensures } \frac{A \log A}{A - I} \geq \frac{B \log B}{B - I}$$

since  $f(t) = \frac{t-1}{\log t}$  ( $t > 0, t \neq 1$ ) and  $f^*(t) = \frac{t \log t}{t-1}$  ( $t > 0, t \neq 1$ ) are both operator monotone functions (see Theorem A underbelow). The function  $f(t) = \frac{t-1}{\log t}$  ( $t > 0, t \neq 1$ ) is said to be “logarithmic function” which is widely used in the theory of heat transfer of the heat engineering and fluid mechanics. Also the function  $f^*(t) = \frac{t \log t}{t-1}$  ( $t > 0, t \neq 1$ ) is said to be “dual logarithmic function”. Related to these two operator inequalities, we shall define the following four orders for strictly positive operators  $A$  and  $B$  such that  $1 \notin \sigma(A), \sigma(B)$ .

**Definition 1.** Let  $A$  and  $B$  be strictly positive operators on a Hilbert space  $H$  such that  $1 \notin \sigma(A), \sigma(B)$ .

(d1) Strictly logarithmic order (denoted by  $A \succ_{sl} B$ ) is defined by  $\frac{A - I}{\log A} > \frac{B - I}{\log B}$ .

(d2) Logarithmic order (denoted by  $A \succ_l B$ ) is defined by  $\frac{A - I}{\log A} \geq \frac{B - I}{\log B}$ .

(d3) Strictly dual logarithmic order (denoted by  $A \succ_{sd} B$ ) is defined by  $\frac{A \log A}{A - I} > \frac{B \log B}{B - I}$ .

(d4) Dual Logarithmic order (denoted by  $A \succ_{dl} B$ ) is defined by  $\frac{A \log A}{A - I} \geq \frac{B \log B}{B - I}$ .

## 2. Simplified proofs of operator monotonicity of logarithmic function and dual logarithmic function

We shall show a direct and simplified proof of the following result [Theorem , 6] and [Example 5.1.12 and Corollary 5.1.11, 5] without use of Löwner general result.

**Theorem A.** *The function  $f$  and  $f^*$  given by*

$$f(t) = \begin{cases} \frac{t-1}{\log t} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}$$

and

$$f^*(t) = \begin{cases} \frac{t \log t}{t-1} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}$$

are operator monotone functions satisfying the symmetry condition:

$$f(t) = tf\left(\frac{1}{t}\right) \text{ and } f^*(t) = tf^*\left(\frac{1}{t}\right).$$

**Proof.** Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . We have only to show the following (i) and (ii) since the latter half is obvious.

- (i) If  $A \geq B$ , then  $\frac{A-I}{\log A} \geq \frac{B-I}{\log B}$ .
- (ii) If  $A \geq B$ , then  $\frac{A \log A}{A-I} \geq \frac{B \log B}{B-I}$ .

First of all, we cite the following obvious result;

- (1)  $T - I = (T^{\frac{1}{n}} - I)(T^{1-\frac{1}{n}} + T^{1-\frac{2}{n}} + \dots + T^{\frac{1}{n}} + I)$  for  $T \geq 0$  and for any natural number  $n$ .
- (2)  $\lim_{n \rightarrow \infty} n(T^{\frac{1}{n}} - I) = \log T$  holds for any  $T \geq 0$ .
- (3) If  $A \geq B \geq 0$ , then  $A^\alpha \geq B^\alpha$  holds for any  $\alpha \in [0, 1]$ . (Löwner-Heinz inequality)

$$\begin{aligned} \text{(i). } \frac{A-I}{n(A^{\frac{1}{n}} - I)} &= \frac{1}{n}(A^{1-\frac{1}{n}} + A^{1-\frac{2}{n}} + \dots + A^{\frac{1}{n}} + I) \text{ by (1) for any natural number } n \\ &\geq \frac{1}{n}(B^{1-\frac{1}{n}} + B^{1-\frac{2}{n}} + \dots + B^{\frac{1}{n}} + I) \text{ by (3) for any natural number } n \\ &= \frac{B-I}{n(B^{\frac{1}{n}} - I)} \text{ for any natural number } n \text{ by (1)} \end{aligned}$$

tending  $n$  to  $\infty$ , so we obtain (i) by (2).

$$\begin{aligned} \text{(ii). } \frac{n(A^{\frac{1}{n}} - I)A}{A-I} &= \frac{n}{(A^{-\frac{1}{n}} + A^{-\frac{2}{n}} + \dots + A^{-1})} \text{ by (1) for any natural number } n \\ &\geq \frac{n}{(B^{-\frac{1}{n}} + B^{-\frac{2}{n}} + \dots + B^{-1})} \text{ by (3) for any natural number } n \\ &= \frac{n(B^{\frac{1}{n}} - I)B}{B-I} \text{ by (1)} \end{aligned}$$

tending  $n$  to  $\infty$ , so we obtain (ii) by (2).

**Remark 1.** It is well known that (i) is equivalent to (ii) in Theorem A. Alternative proof of (i) in the proof of Theorem A is cited in [5]. Related to Theorem A, we remark that the following

result in [Corollary 2.6, 4], [Theorem 2, 7] and [Corollary 5.1.11, 5]: let  $g(t)$  be a continuous positive function such that  $(0, \infty) \rightarrow (0, \infty)$ . Then  $g(t)$  is operator monotone function if and only if  $g^*(t) = \frac{t}{g(t)}$  is operator monotone function. Actually,  $f(t)$  and  $f^*(t)$  in Theorem A satisfy this condition  $f^*(t) = \frac{t}{f(t)}$ .

### 3. Strictly logarithmic order $A \succ_{sl} B$ and logarithmic order $A \succ_l B$

Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . Firstly we shall give Theorem 1 asserting the following

(\*)  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

Secondary, we shall give Corollary 2 showing that there exists an interesting contrast between  $A \geq B > 0$  and  $A > B > 0$  related to  $A \succ_{sl} B$  and  $A \succ_l B$ . Thirdly, we shall give some applications of two characterizations (Theorem A and Theorem B under below) of chaotic order to  $A \succ_{sl} B$  and  $A \succ_l B$  in Corollary 3.

**Lemma 1.** *Let  $A$  and  $B$  be invertible self adjoint operators on a Hilbert space  $H$ . If  $A > B$ , then there exists  $\beta \in (0, 1]$  such that the following inequality holds for all  $\alpha \in (0, \beta)$  ;*

$$\frac{e^{\alpha A} - I}{\alpha A} > \frac{e^{\alpha B} - I}{\alpha B}, \text{ i.e., } e^{\alpha A} \succ_{sl} e^{\alpha B}.$$

**Proof.** There exists  $\varepsilon$  such that  $A - B \geq \varepsilon > 0$ . Choose  $\alpha$  and  $\beta$  such that

$$(4) \quad 0 < \alpha < \text{Min}\left\{\frac{\varepsilon}{2}\left(\frac{e^{\|A\|}}{\|A\|} + \frac{e^{\|B\|}}{\|B\|}\right)^{-1}, 1\right\} = \beta.$$

By an easy calculation, we obtain

$$\begin{aligned} \frac{e^{\alpha A} - I}{A} - \frac{e^{\alpha B} - I}{B} &= \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} A^{n-1} - \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B^{n-1} \\ &= \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1}) \\ &= \frac{\alpha^2}{2!} (A - B) + \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1}) \\ &\geq \frac{\alpha^2}{2!} \varepsilon - \alpha^3 \left[ \sum_{n=3}^{\infty} \frac{1}{n!} (\|A\|^{n-1} + \|B\|^{n-1}) \right] \\ &\geq \alpha^2 \left[ \frac{\varepsilon}{2!} - \alpha \left( \frac{e^{\|A\|}}{\|A\|} + \frac{e^{\|B\|}}{\|B\|} \right) \right] > 0 \quad \text{by (4),} \end{aligned}$$

so that  $\frac{e^{\alpha A} - I}{\alpha A} - \frac{e^{\alpha B} - I}{\alpha B}$  holds, i.e., there exists  $\beta \in (0, 1]$  such that  $e^{\alpha A} \succ_{sl} e^{\alpha B}$  holds for all  $\alpha \in (0, \beta)$  and the proof is complete.

**Theorem 1.** *Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ .*

*If  $\log A > \log B$ , then there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .*

**Proof.** We have only to replace  $A$  by  $\log A$  and also  $B$  by  $\log B$  respectively in Lemma 1.

**Corollary 2.** *Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . Then*

- (i) *If  $A > B > 0$ , then there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .*
- (ii) *If  $A \geq B > 0$ , then  $A^\alpha \succ_l B^\alpha$  holds for all  $\alpha \in (0, 1]$ .*

In Corollary 2, It is interesting to point out the contrast between  $A > B > 0$  and  $A \geq B > 0$ .

**Proof of Corollary 2.** (i). We cite the following obvious and fundamental result (5)

$$(5) \quad \text{If } A > B > 0, \text{ then } \log A > \log B.$$

In fact if  $A > B > 0$ , then  $A \geq B + \varepsilon > B$  for some  $\varepsilon > 0$ , so that  $\log A \geq \log(B + \varepsilon) > \log B$ , that is, (5) holds. (i) follows by (5) and Theorem 1.

(ii). If  $A \geq B > 0$ , then  $A^\alpha \geq B^\alpha$  for all  $\alpha \in (0, 1]$  by Löwner-Heinz inequality and (ii) follows by the result that the function  $f(t) = \frac{t-1}{\log t}$  ( $t > 0, t \neq 1$ ) is an operator monotone function by Theorem A, i.e.,  $f(A^\alpha) \geq f(B^\alpha)$  for all  $\alpha \in (0, 1]$ , so we have (ii).

**Corollary 3.** *Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$  and  $\log A \geq \log B$ . Then*

- (i) *For any  $\delta \in (0, 1]$  there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .*
- (ii) *For any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^{p\alpha} \succ_l B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .*

We cite the following two results in order to give a proof of Corollary 3.

**Theorem A** [1][3]. *Let  $A$  and  $B$  be invertible positive operators on a Hilbert space  $H$ .  $\log A \geq \log B$  holds if and only if for any  $\delta \in (0, 1]$  there exists  $\alpha = \alpha_\delta > 0$  such that  $(e^\delta A)^\alpha \succ B^\alpha$ .*

**Theorem B** [8]. *Let  $A$  and  $B$  be invertible positive operators on a Hilbert space  $H$ .  $\log A \geq \log B$  if and only if for any  $p \geq 0$  there exists a  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^p \geq B^p$ .*

**Proof of Corollary 3.**

(i). As  $\log A \geq \log B$  holds, then for any  $\delta \in (0, 1]$ , there exists  $\alpha' = \alpha'_\delta > 0$  such that  $(e^\delta A)^{\alpha'} > B^{\alpha'}$  by Theorem A. Then  $\log e^\delta A > \log B$  by (5), so that there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$  by Theorem 1.

(ii). As  $\log A \geq \log B$  holds, then for any  $p \geq 0$  there exists a there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^p \geq B^p$  by Theorem B, so we have  $(K_p A)^{p\alpha} \geq B^{p\alpha}$  for all  $\alpha \in (0, 1]$  by (ii) of Corollary 2

**4. Strictly dual logarithmic order  $A \succ_{sdl} B$  and dual logarithmic order  $A \succ_{dl} B$**

Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . Firstly we shall give Theorem 4 asserting the following

(†)  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

Secondary, we shall give Corollary 5 showing that there exists an interesting contrast between  $A \geq B > 0$  and  $A > B > 0$  related to  $A \succ_{sdl} B$  and  $A \succ_{dl} B$ . Thirdly, we shall give some applications of Theorem A and Theorem B to  $A \succ_{sdl} B$  and  $A \succ_{dl} B$  in Corollary 6.

**Lemma 2.** Let  $A$  and  $B$  be invertible self adjoint operators on a Hilbert space  $H$ . If  $A > B$ , then there exists  $\beta \in (0, 1]$  such that the following inequality holds for all  $\alpha \in (0, \beta)$  ;

$$\frac{\alpha A e^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha B e^{\alpha B}}{e^{\alpha B} - I}, \text{ i.e., } e^{\alpha A} \succ_{sdl} e^{\alpha B}.$$

**Proof.** As  $-B > -A$  holds, by applying Lemma 1, there exists  $\beta \in (0, 1]$  such that

$$\frac{e^{-\alpha B} - I}{-\alpha B} > \frac{e^{-\alpha A} - I}{-\alpha A}.$$

holds for all  $\alpha \in (0, \beta)$ . That is,  $\frac{e^{\alpha B} - I}{\alpha B e^{\alpha B}} > \frac{e^{\alpha A} - I}{\alpha A e^{\alpha A}}$  holds iff  $\frac{\alpha A e^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha B e^{\alpha B}}{e^{\alpha B} - I}$  holds, i.e.,

there exists  $\beta \in (0, 1]$  such that  $e^{\alpha A} \succ_{sdl} e^{\alpha B}$  holds for all  $\alpha \in (0, \beta)$  and the proof is complete.

**Theorem 4.** Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ .

If  $\log A > \log B$ , then there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

**Proof.** We have only to replace  $A$  by  $\log A$  and also  $B$  by  $\log B$  respectively in Lemma 2.

**Corollary 5.** Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . Then

- (i) If  $A > B > 0$ , then there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .
- (ii) If  $A \geq B > 0$ , then  $A^\alpha \succ_{dl} B^\alpha$  for all  $\alpha \in (0, 1]$ .

In Corollary 5, It is interesting to point out the contrast between  $A > B > 0$  and  $A \geq B > 0$ .

**Proof of Corollary 5.** By the same way as a proof of Corollary 2, we shall give the following proofs of (i) and (ii).

- (i). If  $A > B > 0$ , then  $\log A > \log B$  holds by (5), so that (i) follows by Theorem 4.
- (ii). If  $A \geq B > 0$ , then  $A^\alpha \geq B^\alpha$  for all  $\alpha \in (0, 1]$  by Löwner-Heinz inequality. The function  $f^*(t) = \frac{t \log t}{t-1}$  ( $t > 0, t \neq 1$ ) is also an operator monotone function by Theorem A, so that  $f^*(A^\alpha) \geq f^*(B^\alpha)$  for all  $\alpha \in (0, 1]$ , so we have (ii).

**Corollary 6.** *Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$  and  $\log A \geq \log B$ . Then*

- (i). *For any  $\delta \in (0, 1]$  there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .*
- (ii). *For any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .*

**Proof of Corollary 6.** We shall obtain Corollary 6 by the same way as one in Corollary 3.

(i). As  $\log A \geq \log B$  holds, then for any  $\delta \in (0, 1]$ , there exists  $\alpha' = \alpha'_\delta > 0$  such that  $(e^\delta A)^{\alpha'} > B^{\alpha'}$  by Theorem A. Then  $\log e^\delta A > \log B$  by (5), so that there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$  by Theorem 4.

(ii). As  $\log A \geq \log B$  holds, then for any  $p \geq 0$  there exists a there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^p \geq B^p$  by Theorem B, so that  $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$  for all  $\alpha \in (0, 1]$  by (ii) of Corollary 5.

**5. An example related to strictly logarithmic order  $A \succ_{sl} B$  and strictly dual logarithmic order  $A \succ_{sdl} B$**

Related to (i) of Corollary 3, we consider the following problem:

(Q1) "Does  $\log A \geq \log B$  ensure that there exists an  $\alpha > 0$  such that  $A^\alpha \succ_l B^\alpha$ ?"

Also related to (i) of Corollary 6, we consider the following problem too;

(Q2) "Does  $\log A \geq \log B$  ensure that there exists an  $\alpha > 0$  such that  $A^\alpha \succ_{dl} B^\alpha$ ?"

In fact, we cite a counterexample to (Q1) and (Q2) as follows.



**Example 1.** Take  $A$  and  $B$  as follows:

$$\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}.$$

Then  $\log A \geq \log B$  holds, but

- (i)  $A^\alpha \succ_l B^\alpha$  does not hold for any  $\alpha > 0$ .
- (ii)  $A^\alpha \succ_{dl} B^\alpha$  does not hold for any  $\alpha > 0$ .
- (iii)  $A^\alpha \geq B^\alpha$  does not hold for any  $\alpha > 0$ .

In fact,  $\log A$  is diagonalized by  $U = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$  as follows;

$$U(\log A)U = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \text{ and } UAU = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^3 \end{pmatrix},$$

so that we have

$$A^\alpha = U \begin{pmatrix} e^{-2\alpha} & 0 \\ 0 & e^{3\alpha} \end{pmatrix} U \text{ and } B^\alpha = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-5\alpha} \end{pmatrix}.$$

Put  $x = e^\alpha > 1$  since  $\alpha > 0$ . At first we show (i). By a slight elaborate calculation, we have

$$\begin{aligned} & \det \left( \frac{A^\alpha - I}{\log A} - \frac{B^\alpha - I}{\log B} \right) \\ &= \begin{vmatrix} \frac{5}{6} - \frac{1}{10x^2} - x + \frac{4x^3}{15} & -\frac{1}{3} + \frac{1}{5x^2} + \frac{2x^3}{15} \\ -\frac{1}{3} + \frac{1}{5x^2} + \frac{2x^3}{15} & \frac{2}{15} + \frac{1}{5x^5} - \frac{2}{5x^2} + \frac{x^3}{15} \end{vmatrix} \\ &= \frac{-1}{50x^7} + \frac{1}{6x^5} - \frac{1}{5x^4} - \frac{4}{25x^2} + \frac{2}{5x} - \frac{3x}{10} + \frac{9x^3}{50} - \frac{x^4}{15} \\ &= \frac{-(x-1)^6(10x^5 + 33x^4 + 48x^3 + 38x^2 + 18x + 3)}{150x^7} < 0 \text{ since } x > 1. \end{aligned}$$

Whence  $A^\alpha \succ_l B^\alpha$  does not hold for any  $\alpha > 0$ , so the proof of (i) is complete.

Next we show (ii). By more elaborate calculation than (i), we obtain

$$\det \left( \frac{A^\alpha \log A}{A^\alpha - I} - \frac{B^\alpha \log B}{B^\alpha - I} \right)$$

$$\begin{aligned}
&= \left| \frac{\frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)}}{\frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)}} \frac{\frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)}}{\frac{3x^4 + 3x^3 + 8x^2 + 8x + 8}{5(x^4 + x^3 - x - 1)} + \frac{5(x - 1)}{-x^6 + x^5 + x - 1}} \right| \\
&= \left( \frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)} \right) \left( \frac{3x^4 + 3x^3 + 8x^2 + 8x + 8}{5(x^4 + x^3 - x - 1)} + \frac{5(x - 1)}{-x^6 + x^5 + x - 1} \right) \\
&\quad - \left( \frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)} \right)^2 \\
&= \frac{-(x - 1)^2(3x^5 + 18x^4 + 38x^3 + 48x^2 + 33x + 10)}{5(x + 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)} < 0 \text{ since } x > 1
\end{aligned}$$

Whence  $A^\alpha \succ_{dl} B^\alpha$  does not hold for any  $\alpha > 0$ , so the proof of (ii) is complete.

Incidentally, we remark that this example also shows that  $\log A \geq \log B$  does not ensure  $A^\alpha \geq B^\alpha$  for any  $\alpha > 0$ . Actually we have

$$\begin{aligned}
&\det(A^\alpha - B^\alpha) \\
&= \left| \frac{\frac{1}{5x^2} - x + \frac{4x^3}{5}}{\frac{2(x^5 - 1)}{5x^2}} \frac{\frac{2(x^5 - 1)}{5x^2}}{\frac{x^8 + 4x^3 - 5}{5x^5}} \right| \\
&= \frac{-1}{5x^7} + x^{-4} - \frac{4}{5x^2} - \frac{4}{5x} + x - \frac{x^4}{5} \\
&= \frac{-(x - 1)^4(x + 1)(x^2 + x + 1)(x^4 + 2x^3 + 4x^2 + 2x + 1)}{5x^7} < 0 \text{ since } x > 1,
\end{aligned}$$

that is,  $A^\alpha \geq B^\alpha$  does not hold for any  $\alpha > 0$ , so (iii) is shown. In [2], there is another nice example that  $\log A \geq \log B$  does not ensure  $A^\alpha \geq B^\alpha$  for any  $\alpha > 0$ . In fact, we construct Example 1 inspired by an excellent method in [2].

## 6. Concluding remarks

Let  $A$  and  $B$  be strictly positive operators such that  $1 \notin \sigma(A), \sigma(B)$ . We can obtain the following interesting contrast among  $A > B > 0$ ,  $A \geq B > 0$ ,  $\log A > \log B$  and  $\log A \geq \log B$  by summarizing our results in this paper.

( $\star$ )  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

( $\dagger$ )  $\log A > \log B \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

( $l$ -i)  $A > B > 0 \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(*l*-ii)  $A \geq B > 0 \implies A^\alpha \succ_l B^\alpha$  for all  $\alpha \in (0, 1]$ .

(*l*-iii)  $\log A \geq \log B \implies$  for any  $\delta \in (0, 1]$ , there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(*l*-iv)  $\log A \geq \log B \implies$  for any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p A)^{p\alpha} \succ_l B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .

(*dl*-i)  $A > B > 0 \implies$  there exists  $\beta \in (0, 1]$  such that  $A^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(*dl*-ii)  $A \geq B > 0 \implies A^\alpha \succ_{dl} B^\alpha$  for all  $\alpha \in (0, 1]$ .

(*dl*-iii)  $\log A \geq \log B \implies$  for any  $\delta \in (0, 1]$ , there exists  $\beta = \beta_\delta \in (0, 1]$  such that  $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$  holds for all  $\alpha \in (0, \beta)$ .

(*dl*-iv)  $\log A \geq \log B \implies$  for any  $p \geq 0$  there exists  $K_p > 1$  such that  $K_p \rightarrow 1$  as  $p \rightarrow +0$  and  $(K_p)^{p\alpha} \succ_{dl} B^{p\alpha}$  for all  $\alpha \in (0, 1]$ .

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## 7. Appendix

### Simple proof of the concavity on operator entropy $f(A) = -A \log A$

A capital letter means a bounded linear and *strictly positive* operator on a Hilbert space. Here we shall give a simple proof of the following well known and excellent result obtained by [1] and [2] independently.

**Theorem A.**  $f(A) = -A \log A$  is concave function for any  $A > 0$ .

**Proof.** Firstly we recall the following obvious result

$$(*) \quad \lim_{n \rightarrow \infty} (T^{\frac{1}{n}} - I)n = -\log T \quad \text{for any } T > 0.$$

As  $g(t) = t^q$  is operator concave for  $q \in [0, 1]$ , then for  $A > 0$ ,  $B > 0$  and  $\alpha, \beta \in [0, 1]$  such that  $\alpha + \beta = 1$

$$(\alpha A + \beta B)^{1 - \frac{1}{n}} \geq \alpha A^{1 - \frac{1}{n}} + \beta B^{1 - \frac{1}{n}} \quad \text{for any natural number } n$$

so we obtain

$$(\alpha A + \beta B) \left( (\alpha A + \beta B)^{-\frac{1}{n}} - I \right) n \geq \alpha A (A^{-\frac{1}{n}} - I)n + \beta B (B^{-\frac{1}{n}} - I)n$$

tending  $n \rightarrow \infty$ , we have

$$-(\alpha A + \beta B) \log(\alpha A + \beta B) \geq (-\alpha A \log A - \beta B \log B) \quad \text{by } (*)$$

that is,

$$f(\alpha A + \beta B) \geq \alpha f(A) + \beta f(B)$$

so the proof is complete.

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